

# Bipartite entanglement as an aspect of pure spinor geometry

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## Abstract

Relying on the mathematical analogy of the pure states of a two-qubit system with four-component Dirac spinors, we provide an alternative consideration of quantum entanglement using the mathematical formulation of Cartan's pure spinors. A result of our analysis is that the Cartan equation of two qubits state is entanglement sensitive in a way that the Dirac equation for fermions is mass sensitive. The Cartan equation for unentangled qubits is reduced to a pair of Cartan equations for single qubits as the Dirac equation for massless fermions separates into two Weyl equations. Finally, we establish a correspondance between the separability condition in qubit geometry and the separability condition in spinor geometry.

## 1 Introduction

Although in 1932 von Neumann had completed basic elements of nonrelativistic quantum description of the world, it were Einstein, Podolsky and Rosen (EPR) and Schödinger who first recognized a “spooky” feature of quantum machinery which lies at center of interest of physics of XXI century. This feature, known as entanglement [1], implies the existence of global states of composite system which cannot be written as a product of the states of individual subsystems [2]. The definition of entanglement relies on the tensor-product structure of the state-space of a composite quantum system [3]. Unfortunately such a tensor product structure is not present in a large class of systems of major physical interest: ensembles of indistinguishable particles [4]. Recently an algebraic approach to quantum non - separability was applied to the case of two qubits, based on the partition of the algebra of observables into independent subalge-

ras [5]. In this paper we shall tackle quantum entanglement from a geometric perspective.

The concept of quantum entanglement is believed to play an essential role in quantum information processing. Due to the fundamental significance of quantum information theory, many efforts have been devoted to the characterization of entanglement in terms of several physically equivalent measures, such as entropy and concurrence [6, 7]. One of the most fundamental and most studied models of quantum information theory is the two-qubit (two-spin- $\frac{1}{2}$ ) system (see, e.g., [8]). The pure states of the two-qubit system are described in terms of vectors of a four-dimensional Hermitian (complex) vector space  $\mathbb{C}_4$ . It is of interest to describe their quantum evolution in a suitable representation space, in order to get some insight into the subtleties of this complicated problem. A well known tool in quantum optics for the single qubit is the Bloch sphere representation, where the simple qubit state is faithfully represented, up to its overall phase, by a point on a standard sphere  $S^2$ , whose coordinates are expectation values of physically interesting operators for the given quantum state. Guided by the relation between the Bloch sphere and a geometric object called the Hopf fibration of the  $S^3$  hypersphere [11], a generalization for a two-qubit system was proposed [12], in the framework of the (high dimensional)  $S^7$  sphere Hopf fibration, and will be recalled below.

In the last few years the application of quantum information principles and viewpoints within relativistic quantum mechanics and quantum field theory has begun to be explored [9, 10]. In the realm of relativistic quantum mechanics and quantum field theory in 3+1-dimensional Minkowski space-time fermions, an essential part of physical reality, are mainly described by means of Dirac spinors – elements of a four-component complex vector space which are transforming under the spin representation of the Poincaré group (inhomogeneous Lorentz group) of space-time. Relying on this mathematical analogy of the pure states of the two-qubit system of quantum information theory with four-component spinors we introduce the concept of spinors entanglement and explore the connection between one of the most fundamental models of relativistic quantum mechanics and one of the most fundamental models of quantum information theory. In this paper certain aspects of this problem are studied. The same problem, but from a different viewpoint, is addressed by Scharnhorst [13].

As far as computation is concerned, the pure spinors provide us an elegant way to put the calculations into a compact form, and have (by nature) a natural geometrical interpretation. Spinors were first used under such a name by physicists in the field of quantum mechanics. In most of their general mathematical forms, spinors were discovered in 1913 by Elie Cartan [14], who conjectured that the fundamental geometry appropriate for the description and understanding of elementary natural phenomena is spinor geometry, more precisely the geometry of simple spinors, later named pure by C. Chevalley [15], rather than the one of euclidean vectors which may be constructed bilinearly from spinors. A fundamental property of pure spinors is that of generating null vectors in momentum spaces, where the Cartan equations defining pure spinors are identical to equations of motion for massless systems [16]. In this paper, we find that

a pure spinor, naturally generating Bloch vectors, may be associated with a simple qubit state. A result of our analysis is that the Cartan's equation of two qubits states is entanglement sensitive in a similar way to the Dirac equation, for massless fermions, separates into two equations (the Cartan-Weyl equations).

The paper is organised as follows. In section 2, using the Bloch sphere representation, we derive the Cartan's equations of a single qubit state. In section 3 we introduce the technique by which Cartan's equations of a two-qubits state are derived. This leads to a discussion of the role of entanglement in structure of pure spinor geometry. In section 4 we consider the separability conditions of two qubits in pure spinor geometry. We establish, in section 5, a correspondance between the separability condition in qubit geometry and the separability condition in spinor geometry. We conclude in section 6 with a brief discussion.

## 2 The Cartan equation of single qubit state

A single qubit state reads

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1. \quad (1)$$

In the spin  $\frac{1}{2}$  context, the orthonormal basis  $\{|0\rangle, |1\rangle\}$  is composed of the two eigenvectors of the  $\sigma_z$  Pauli spin operator. A convenient way to represent  $|\psi\rangle$  (up to a global phase) is provided by the Bloch sphere. The set of states  $\exp i\varphi |\psi\rangle$ , with  $\varphi \in [0, 2\pi[$ , is mapped onto a point on  $S^2$  (the usual sphere in  $\mathbb{R}^3$ ):

$$X_1^2 + X_2^2 + X_3^2 = 1 \quad (2)$$

with coordinates

$$\begin{aligned} X_1 &= \langle\psi|\sigma_1|\psi\rangle = 2\text{Re}(a^*b), \\ X_2 &= \langle\psi|\sigma_2|\psi\rangle = 2\text{Im}(a^*b), \\ X_3 &= \langle\psi|\sigma_3|\psi\rangle = |a|^2 - |b|^2, \end{aligned} \quad (3)$$

( $a^*$  is the complex conjugate of  $a$ ) to be the expectation values of physically interesting operators for the given quantum state. Here, the  $\sigma_{1,2,3}$  are the standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)$$

Recall the relation between Bloch sphere coordinates and the pure state density matrix  $\rho_{|\psi\rangle}$ :

$$\rho_{|\psi\rangle} = \rho_{\exp i\varphi|\psi\rangle} = |\psi\rangle\langle\psi| = \frac{1}{2} \begin{pmatrix} 1 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & 1 - X_3 \end{pmatrix}. \quad (5)$$

The vanishing determinant of  $\rho$  is related to the unit radius of the Bloch sphere. For single qubit mixed states, the latter condition is relaxed and density matrices

are in one-to-one correspondance with points in the Bloch “ball”, the interior of the pure state Bloch sphere. Thus giving a Bloch ball  $B^3$  in  $\mathbb{R}^3$ ,

$$X_1^2 + X_2^2 + X_3^2 = R^2 \quad (6)$$

any point  $(X_1, X_2, X_3)$  with  $|R| \leq 1$  corresponds to a valid qubit state. Points on the surface ( $|R| = 1$ ) correspond to pure qubit states. Points with  $|R| \leq 1$  correspond to mixed states (the origin  $|R| = 0$  corresponds to completely mixed state).

We now proceed to the linearization of eq.(6). Our procedure is exactly the same as followed by Dirac in his classical paper on the spinning electron[17] and is based on real Clifford algebras [18]. The Bloch vector, which defines the Bloch ball (6) and lives in vector space  $V = \mathbb{R}^3$ , is associated with Clifford algebra  $Cl(3)$ , generated by the Pauli matrices  $\sigma_1, \sigma_2, \sigma_3$ , and can be written as

$$X = X_i \sigma^i. \quad (7)$$

Following Cartan we may define a spinor  $|\psi\rangle$  with the equation (terms with repeated indices are meant to be summed):

$$(X_i \sigma^i - R) |\psi\rangle = 0, \quad i = 1, 2, 3. \quad (8)$$

This equation is the linearised version of the quadratic equation (6). Notice that multiplying (8) from the left with  $\langle\psi| = |\psi\rangle^\dagger$  and using the normalization condition for  $|\psi\rangle$  and eqs.(3) we obtain the Bloch ball (6).

### 3 The Cartan equation of two-qubits state

Now consider two “separable” single qubits, let say Alice:

$$|\psi^A\rangle = a |0\rangle + b |1\rangle, \quad a, b \in \mathbb{C} \quad (9)$$

represented in Bloch ball:

$$(X_1^A)^2 + (X_2^A)^2 + (X_3^A)^2 = (X_0^A)^2 \quad (10)$$

and Bob:

$$|\psi^B\rangle = c |0\rangle + d |1\rangle, \quad c, d \in \mathbb{C} \quad (11)$$

represented in Bloch ball:

$$(X_1^B)^2 + (X_2^B)^2 + (X_3^B)^2 = (X_0^B)^2. \quad (12)$$

If we exploit the fact that the Bloch sphere coordinates are defined uniquely up to sign, the associated Cartan’s equations take the form:

$$\begin{aligned} (X_i^A \sigma^i - X_0^A) |\psi^A\rangle &= 0 \\ (X_i^B \sigma^i + X_0^B) |\psi^B\rangle &= 0 \end{aligned} \quad (13)$$

respectively.

Equations (13) may be expressed as a single equation for the four component Dirac spinor  $|\Psi\rangle = |\psi^A\rangle \oplus |\psi^B\rangle$ . Indicating the matrices  $\gamma^\mu$  as

$$\gamma^\mu : \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}, \quad (14)$$

with

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (15)$$

and

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \quad (16)$$

its volume element, we may write (13) in the form

$$X_\mu^{A,B} \gamma^\mu (1 \pm \gamma^5) |\Psi\rangle = 0 \quad (17)$$

where  $|\psi^{A,B}\rangle = \frac{1}{2} (1 \pm \gamma^5) |\Psi\rangle$  the Weyl spinors and eigenspinors of  $\gamma^5$ . Writing the Hermitean conjugate of eq.(17):

$$X_\mu^{A,B} \langle \bar{\Psi} | \gamma_\mu (1 \pm \gamma^5) = 0 \quad (18)$$

where  $\langle \bar{\Psi} | = \langle \Psi | \gamma^0$ , we may define the quantities  $X_\mu^{A,B}$  by the relations<sup>1</sup>

$$X_\mu^{A,B} = \langle \bar{\Psi} | \gamma_\mu \left( \frac{1 \pm \gamma^5}{2} \right) |\Psi\rangle. \quad (19)$$

It is not hard to check that:

$$X_\mu^{A,B} X_{A,B}^\mu = 0 \rightarrow \left( X_1^{A,B} \right)^2 + \left( X_2^{A,B} \right)^2 + \left( X_3^{A,B} \right)^2 = \left( X_0^{A,B} \right)^2 \quad (20)$$

which means that the two qubits remain non-entangled, associated with two independent Bloch spheres.

### 3.1 Left-handed and right-handed Bloch spheres

Writing the two qubits in this way, an interesting aspect of the corresponding Bloch spheres is revealed. An arbitrary unitary operation on the single qubit can be represented by a unitary matrix on  $\mathbb{C}^2$ , a member of  $SU(2)$  group. The same operation can be also written with a rotation matrix, from  $SO(3)$  group, of the Bloch sphere by an angle  $\omega_{ij}$  about axis defined by bivector  $\sigma^i \wedge \sigma^j$ . Thus

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<sup>1</sup>remember that  $\left( \frac{1 \pm \gamma^5}{2} \right)^2 = \left( \frac{1 \pm \gamma^5}{2} \right)$

under an orthogonal transformation  $L_i^j \in SO(3)$  of Bloch sphere coordinates a single qubit state transforms according to

$$|\psi\rangle' (L_i^j X^i) = \Lambda |\psi\rangle (X^j) \quad (21)$$

where

$$\Lambda = \exp\left(\frac{1}{2}\omega_{ij}W^{ij}\right) = \exp\left(\frac{i}{8}\omega_{ij}[\sigma^i, \sigma^j]\right) \quad (22)$$

the invertible elements that define  $Spin(3)$  group, the double cover of  $SO(3)$ .

Writing the two single non-entangled qubits in terms of Weyl spinors, eq.(17), as we have seen, the associated Clifford algebra is generated by  $\gamma^\mu$  satisfying (15). This representation is reducible. Specially, in block notations let

$$\gamma^\mu = \begin{pmatrix} 0 & \zeta^\mu \\ \theta^\mu & 0 \end{pmatrix}. \quad (23)$$

Then  $2 \times 2$  matrices  $\eta^\mu$  and  $\theta^\mu$  satisfy

$$\zeta^\mu \theta^\nu + \zeta^\nu \theta^\mu = 2\eta^{\mu\nu}, \quad \theta^\mu \zeta^\nu + \theta^\nu \zeta^\mu = 2\eta^{\mu\nu}. \quad (24)$$

The matrices  $M_L^{\mu\nu} \equiv \frac{i}{4}[\zeta^\mu, \theta^\nu]$  and  $M_R^{\mu\nu} \equiv \frac{i}{4}[\theta^\mu, \zeta^\nu]$  generate different rotations in the  $\mu - \nu$  plane. The group thus has two “pieces”

$$\begin{aligned} \Lambda_L &= \exp\left(\frac{1}{2}\omega_{\mu\nu}M_L^{\mu\nu}\right), \\ \Lambda_R &= \exp\left(\frac{1}{2}\omega_{\mu\nu}M_R^{\mu\nu}\right). \end{aligned} \quad (25)$$

They are topologically disjunct (disjoint), and there is no continuous path from one piece to the other. So we can define two types of spinors besides two kinds of generators  $\eta^\mu$  and  $\theta^\mu$ . Under an arbitrary unitary operation the pair of qubits  $|\psi^A\rangle$  and  $|\psi^B\rangle$  transforms as

$$\begin{aligned} |\Psi\rangle' &= \exp\left(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right) = \exp\left(\frac{i}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]\right) |\Psi\rangle \\ \begin{pmatrix} |\psi^A\rangle' \\ |\psi^B\rangle' \end{pmatrix} &= \begin{pmatrix} \Lambda_L |\psi^A\rangle \\ \Lambda_R |\psi^B\rangle \end{pmatrix} \end{aligned} \quad (26)$$

identifying Alice qubit as left-handed Weyl spinor and Bob qubit as right-handed Weyl spinor. So Cartan-Weyl equations (17) define two types of qubit Bloch spheres, a left-handed and a right-handed Bloch sphere in correspondence to the left-handed and right-handed Weyl spinor.

### 3.2 Mixing terms and entanglement

We now proceed one step further and take the sum of the components of the two Bloch spheres

$$X_\mu^A + X_\mu^B = X_\mu = \langle \bar{\Psi} | \gamma_\mu | \Psi \rangle, \quad (27)$$

where  $|\Psi\rangle = |\psi^A\rangle \oplus |\psi^B\rangle$ . This means that starting from two pure spinor spaces we operated the most obvious operation: their direct sum spanned by spinors of double dimension. The corresponding operation in vector space, represented by eq.(27), defines vector  $X_\mu$ , with components:

$$\begin{aligned} X_1 &= \langle \bar{\Psi} | \gamma_1 | \Psi \rangle = a^*b + b^*a - c^*d - d^*c \\ X_2 &= \langle \bar{\Psi} | \gamma_2 | \Psi \rangle = i(-a^*b + b^*a + c^*d - d^*c) \\ X_3 &= \langle \bar{\Psi} | \gamma_3 | \Psi \rangle = |a|^2 - |b|^2 - |c|^2 + |d|^2 \\ X_0 &= \langle \bar{\Psi} | \gamma_0 | \Psi \rangle = -|a|^2 - |b|^2 - |c|^2 - |d|^2 \end{aligned} \quad (28)$$

which is non null:

$$X_1^2 + X_2^2 + X_3^2 - X_0^2 = -C^2. \quad (29)$$

It has been shown [16] that given two pure spinors  $|\psi^{A,B}\rangle$  and the corresponding null vectors  $X_\mu^{A,B} \in \mathbb{R}^4$  with components

$$X_\mu^{A,B} = \langle \bar{\Psi} | \gamma_\mu \left( \frac{1 \pm \gamma^5}{2} \right) | \Psi \rangle \quad (30)$$

their sum eq.(27) is a projection on  $\mathbb{R}^4$  of a null vector  $X_l \in \mathbb{R}^6$  which is obtained by adding to  $X_\mu$  the two extra components

$$\begin{aligned} X_4 &= \langle \bar{\Psi} | \gamma_4 | \Psi \rangle = i \langle \bar{\Psi} | \Psi \rangle = i(a^*c + b^*d - c^*a - d^*b), \\ X_5 &= \langle \bar{\Psi} | \gamma_5 | \Psi \rangle = -a^*c - b^*d - c^*a - d^*b. \end{aligned} \quad (31)$$

It is easy to verify that

$$X_4^2 + X_5^2 = C^2, \quad (32)$$

thus eq.(29) takes the form of a generalized Bloch sphere

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 - X_0^2 = 0. \quad (33)$$

As for the standard Bloch sphere case, the  $X_l$  coordinates are also expectation values of simple operators in the two-qubits state. With  $X_l$  we may generate the Cartan's equation

$$(X_\mu \gamma^\mu + iX_4 + X_5 \gamma^5) |\Psi\rangle = 0, \quad (34)$$

or in block notation

$$\begin{pmatrix} (iX_4 + X_5) \mathbf{1}_2 & (X_i \sigma^i + X_0) \mathbf{1}_2 \\ (X_i \sigma^i - X_0) \mathbf{1}_2 & (iX_4 - X_5) \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} |\psi^A\rangle \\ |\psi^B\rangle \end{pmatrix} = 0 \Rightarrow \begin{cases} (X_i \sigma^i + X_0) |\psi^B\rangle = -(X_5 + iX_4) |\psi^A\rangle \\ (X_i \sigma^i - X_0) |\psi^A\rangle = (X_5 - iX_4) |\psi^B\rangle \end{cases} \quad (35)$$

Let us express the complex vector  $X_5 + iX_4$  in the polar form:

$$X_5 \pm iX_4 = M \exp\left(\pm i \frac{\omega}{2}\right) \quad (36)$$

Substituting this in eq.(35) and multiplying the second by  $\exp(i\frac{\omega}{2})$  we obtain:

$$\begin{cases} (X_i\sigma^i + X_0) |\psi^B\rangle = -M e^{i\frac{\omega}{2}} |\psi^A\rangle \\ (X_i\sigma^i - X_0) e^{i\frac{\omega}{2}} |\psi^A\rangle = M |\psi^B\rangle \end{cases} \quad (37)$$

We see that  $|\psi^A\rangle$  appears with a phase factor  $e^{i\frac{\omega}{2}}$  corresponding to a rotation through an angle  $\omega$  in the circle

$$X_4^2 + X_5^2 = M^2. \quad (38)$$

Note that setting

$$X_4 = X_5 = 0 \rightarrow M = 0, \quad (39)$$

eqs.(35) reduce to

$$\begin{cases} (X_i\sigma^i + X_0) |\psi^B\rangle = 0 \\ (X_i\sigma^i - X_0) |\psi^A\rangle = 0 \end{cases} \quad (40)$$

The Hilbert space for two non-entangled qubits A and B is expected to be the product of two 2-dimensional spheres  $S_A^2 \times S_B^2$ , each sphere being the Bloch sphere associated with the given qubit. This property is clearly displayed here with the re-production of the Cartan-Weyl equations (13). Equations (35) can then be written in the form

$$\begin{cases} (X_i^B\sigma^i + X_0^B) |\psi^B\rangle = -M e^{i\frac{\omega}{2}} |\psi^A\rangle \\ (X_i^A\sigma^i - X_0^A) e^{i\frac{\omega}{2}} |\psi^A\rangle = M |\psi^B\rangle \end{cases} \quad (41)$$

indicating that  $M$  mixes the two single qubit states  $|\psi^{A,B}\rangle$  into an entangled two qubit state  $|\Psi\rangle$ , in a similar way to the mixing of the left- and right-chiral Weyl spinors by mass terms into a propagating Dirac spinor. In geometrical terms, entanglement arises as the term which does not allow a generalized Bloch sphere to be given from the sum of two Bloch spheres coordinates.

## 4 “Separability” condition of generalized Bloch sphere transformations

Under an orthogonal transformation  $L_l^k \in SO(5, 1)$  of generalized Bloch sphere coordinates a two-qubits state  $|\Psi\rangle$  transforms according to

$$|\Psi\rangle' (L_l^k X^l) = \Lambda |\Psi\rangle (X^k), \quad k, l = 0, 1, 2, 3, 4, 5 \quad (42)$$

where

$$\Lambda = \exp\left(\frac{1}{2}\omega_{kl}\mathcal{M}^{kl}\right) = \exp\left(\frac{i}{8}\omega_{kl}[\gamma^k, \gamma^l]\right) \quad (43)$$

the invertible elements that define  $Spin(5, 1)$  group, the double cover of  $SO(5, 1)$ . The generators  $\mathcal{M}^{kl}$ , of rotations in  $Spin(5, 1)$ , satisfy the Lie algebra of  $SO(5, 1)$ :

$$[\mathcal{M}^{kl}, \mathcal{M}^{mn}] = i(\mathcal{M}^{kn}\mathbf{h}^{lm} + \mathcal{M}^{lm}\mathbf{h}^{kn} - \mathcal{M}^{km}\mathbf{h}^{ln} - \mathcal{M}^{ln}\mathbf{h}^{km}) \quad (44)$$



with  $h = \text{diag}(-, +, +, +, +, +)$  the corresponding metric. The algebra of these generators cannot decouple into two algebras, one for each set of generators associated with single qubits.

Let us set

$$X_4 = X_5 = 0 \quad (45)$$

in order to obtain the projection:

$$X_1^2 + X_2^2 + X_3^2 - X_0^2 = 0 \quad (46)$$

of generalized Bloch sphere (33). In that case, as we have seen, the generators  $\mathcal{M}^{kl}$  reduce to

$$M^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu], \quad \mu, \nu = 0, 1, 2, 3 \quad (47)$$

and satisfy the Lie algebra of  $SO(3, 1)$ :

$$[M^{\kappa\lambda}, M^{\mu\nu}] = i (M^{\kappa\nu} h^{\lambda\mu} + M^{\lambda\mu} h^{\kappa\nu} - M^{\kappa\mu} h^{\lambda\nu} - M^{\lambda\nu} h^{\kappa\mu}) \quad (48)$$

with  $h = \text{diag}(-, +, +, +)$  the reduced metric. Due to the reducible representation of generators  $\gamma^\mu$  (see 3.1),  $M^{\mu\nu}$  split into two pieces

$$M_L^{\mu\nu} \equiv \frac{i}{4} [\zeta^\mu, \theta^\nu] \text{ and } M_R^{\mu\nu} \equiv \frac{i}{4} [\theta^\mu, \zeta^\nu]. \quad (49)$$

We have to show now that these two generators decouple into two  $SU(2)$  algebras, one for each single qubit. Starting with generator  $M_L^{\mu\nu}$ , one can explicitly separates it into two generators

$$\begin{aligned} J_i^L &= \frac{1}{2} \varepsilon_{ijk} M_{jk}^L = \frac{1}{2} \sigma_i \\ K_i^L &= M_{0i}^L = -\frac{i}{2} \sigma_i \end{aligned} \quad (50)$$

which are respectively the generator of rotations of Bloch sphere and of Bloch sphere resizing. Remember that the Bloch ball representation of a mixed state was recalled, where the set of pure states forms the  $S^2$  boundary, and the center of the Bloch ball corresponds to maximally mixed states. We define the combinations of these two sets of generators

$$\begin{aligned} \mathcal{L}_i &= \frac{1}{2} (J_i^L + i K_i^L) = \frac{1}{2} \sigma_i \\ \bar{\mathcal{L}}_i &= \frac{1}{2} (J_i^L - i K_i^L) = 0 \end{aligned} \quad (51)$$

and find

$$\begin{aligned} [\mathcal{L}_i, \mathcal{L}_j] &= i \varepsilon_{ijk} \mathcal{L}_k \\ [\mathcal{L}_i, \bar{\mathcal{L}}_j] &= 0 \end{aligned} \quad (52)$$

Similarly with generators  $M_R^{\mu\nu}$  we obtain

$$\begin{aligned}\mathcal{R}_i &= \frac{1}{2} (J_i^R + iK_i^R) = 0 \\ \bar{\mathcal{R}}_i &= \frac{1}{2} (J_i^R - iK_i^R) = \frac{1}{2}\sigma_i\end{aligned}\tag{53}$$

which satisfy the commutation relations

$$\begin{aligned}[\bar{\mathcal{R}}_i, \bar{\mathcal{R}}_j] &= i\varepsilon_{ijk}\bar{\mathcal{R}}_k \\ [\mathcal{R}_i, \mathcal{R}_j] &= 0.\end{aligned}\tag{54}$$

Therefore the generators, associated with generalized Bloch sphere, can be split into two subsets, which commute with each other and each satisfy the same commutation relations as the group  $SU(2)$ , provided that the number of generalized Bloch sphere dimensions is reduced by two.

## 5 Mixing in qubit geometry

The Hilbert space  $\mathcal{E}$  for the system of two qubits is the tensor product of the individual Hilbert spaces  $\mathcal{E}_1 \otimes \mathcal{E}_2$ , with a direct product basis  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ . Thus, the two qubit state  $|\Psi\rangle$  reads

$$|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\tag{55}$$

with

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.\tag{56}$$

$|\Psi\rangle$  is said “separable” if, at the price of possible basis changes in  $\mathcal{E}_1$  and  $\mathcal{E}_2$  separately, it can be written as a single product. As is well known, the separability condition reads:

$$ad - bc = 0.\tag{57}$$

The aim of this paragraph is to establish a correspondance between this well-known separability condition (57) in qubit geometry and the non mixing criteria (39) in spinor geometry we obtained at the previous paragraph. Let us start with the generalized Bloch sphere

$$\mathcal{X}_1^2 + \mathcal{X}_2^2 + \mathcal{X}_3^2 + \mathcal{X}_4^2 + \mathcal{X}_5^2 = R^2\tag{58}$$

which provides a representation (up to a global phase) not of  $|\Psi\rangle$ , but of:

$$|\tilde{\Psi}\rangle := \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} |\Psi\rangle = \begin{pmatrix} a^* \\ b \\ c^* \\ d \end{pmatrix},\tag{59}$$

where  $J$  an operator which takes the complex conjugate of a complex number [12]. We may consider quadratic form (58) as a non-null euclidean vector which

lives in vector space  $\mathbb{R}^5$  and is associated to Clifford algebra  $\mathcal{Cl}(5)$ . Choosing an appropriate representation of its generators,  $\mathcal{G}_t$ :

$$\begin{aligned}\mathcal{G}_1 = \sigma_1 \otimes \mathbf{1}_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \mathcal{G}_2 = \sigma_2 \otimes \sigma_3 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \\ \mathcal{G}_3 = \sigma_3 \otimes \mathbf{1}_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \mathcal{G}_4 = \sigma_2 \otimes \sigma_1 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\ \mathcal{G}_5 = \sigma_2 \otimes \sigma_2 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\end{aligned}\tag{60}$$

satisfying:

$$\{\mathcal{G}_t, \mathcal{G}_p\} = 2\delta_{tp},\tag{61}$$

we take the Cartan equation

$$(\mathcal{G}^t \mathcal{X}_t - R) \left| \tilde{\Psi} \right\rangle = 0\tag{62}$$

with the components to be equal to:

$$\begin{aligned}R &= \left\langle \tilde{\Psi} \left| \tilde{\Psi} \right\rangle = |a|^2 + |b|^2 + |c|^2 + |d|^2 \\ \mathcal{X}_1 &= \left\langle \tilde{\Psi} \left| \mathcal{G}_1 \right| \tilde{\Psi} \right\rangle = 2 \operatorname{Re} (ab + d^* c^*) \\ \mathcal{X}_2 &= \left\langle \tilde{\Psi} \left| \mathcal{G}_2 \right| \tilde{\Psi} \right\rangle = 2 \operatorname{Im} (ab + d^* c^*) \\ \mathcal{X}_3 &= \left\langle \tilde{\Psi} \left| \mathcal{G}_3 \right| \tilde{\Psi} \right\rangle = |a|^2 - |b|^2 + |c|^2 - |d|^2 \\ \mathcal{X}_4 &= \left\langle \tilde{\Psi} \left| \mathcal{G}_4 \right| \tilde{\Psi} \right\rangle = 2 \operatorname{Im} (cb - ad) \\ \mathcal{X}_5 &= \left\langle \tilde{\Psi} \left| \mathcal{G}_5 \right| \tilde{\Psi} \right\rangle = 2 \operatorname{Re} (cb - ad)\end{aligned}\tag{63}$$

Note that the separability condition (57) implies

$$\mathcal{X}_4 = \mathcal{X}_5 = 0,\tag{64}$$

thus, indeed, the dimensions of two qubits Bloch sphere are entanglement sensitive. The dependence of generalized Bloch sphere on entanglement was also obtained from a different framework of derivation [12]. In that work the two-qubit entanglement has been studied by making appeal to quaternions and was found that an operator (similar to ours), termed “entangler”, provides an estimate for the entanglement. Such an operator has been already largely used in quantifying entanglement [6], with a quantity, called “concurrence”.

## 6 Discussion and conclusions

In this paper we attempted to show how the elegant geometry of pure spinors could be helpful for throwing some light on tantalizing notion of quantum entanglement. Our main goal was, taking advantage of the mathematical analogy of the pure states of a two qubits system with four-component Dirac spinors, to provide an alternative consideration of quantum entanglement using the mathematical formulation of Cartan's pure spinors. The understanding of the concept of entanglement applied here is in various respect a generalized one and differs from the one conventionally applied in the analysis of two spatially separated spins.

A single qubit state may be represented by a two-component Pauli spinor and, from this, if pure, vectors of null quadrics in pseudo euclidean spaces or Bloch vectors in our case, may be constructed. It is then natural to try to imbed spinor spaces, and the corresponding constructed Bloch spheres, in higher dimensional ones in order to derive quantum entanglement. Starting from two pure spinor spaces we operated the most obvious operation: their direct sum spanned by spinors of double dimension, gave rise to a higher dimensional pure spinor space. The corresponding generalized null vectors, which define the generalized Bloch sphere, is given by the sum of the corresponding Bloch vectors and are obtained by adding two extra components. These two components provide an entanglement measure, in the sense that setting these two components equal to zero the generalized Bloch sphere reduces in two Bloch spheres representing the two single qubit states.

The present explorative study based on the formal mathematical analogy of four-component spinors with the pure states of a two qubits system leaves many questions open for further research. It has been already suggested that spinors provide us with a link between space-time and entanglement properties of spinors [19]. We leave for a future publication a systematic treatment of multipartite entanglement using pure spinors geometry. This way entanglement properties of spinors have possibly a relation to the problem of emerging large number of space dimensions, a subject worth of further study.

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